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IMPLEMENTATION OF CLAMPED AND SIMPLY SUPPORTED BOUNDARY CONDITIONS IN THE GDQ FREE VIBRATION ANALYSIS OF BEAMS AND PLATES

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Abstract—In this paper, a new methodology for implementing the clamped and simply supported boundary conditions is presented for the free vibration analysis of beams and plates using the generalized differential quadrature (GDQ) method. The proposed approach directly substitutes the boundary conditions into the governing equations and is referred to as SBCGE approach. The SBCGE approach is presented to overcome the drawbacks of previous approaches in treating the boundary conditions. A comparison of the SBCGE approach with the method of modifying weighting coefficient matrices (MWCM) is made by their application to the vibration analysis of beams and plates with combinations of simply supported and clamped boundary conditions. Some details of the GDQ method are also described in the paper.

1. INTRODUCTION

Most numerical simulations of engineering problems can be currently carried out by conventional low order finite differences and finite elements using a large number of grid points. However, in some practical applications, the numerical solutions of partial differential equations are required at only a few specified points in the physical domain. For acceptable accuracy, the conventional low order techniques still require the use of a large number of grid points to obtain accurate solutions at these specified points. In seeking a more efficient method using just a few grid points to obtain accurate numerical results, the technique of differential quadrature (DQ) was proposed by Bellman *et al.* (1972). The DQ method follows the concept of classical integral quadrature. DQ approximates a spatial derivative of a function with respect to a coordinate at a discrete point as a weighted linear sum of all the functional values in the whole domain of that coordinate direction. The key to DQ is to determine the weighting coefficients for any order derivative discretization. Bellman *et al.* (1972) suggested two methods to determine the weighting coefficients of the first order derivative. The first method solves an algebraic equation system. The second uses a simple algebraic formulation, but with the coordinates of grid points chosen as the roots of the shifted Legendre polynomial. Most previous applications of DQ in engineering (see, e.g., Bellman *et al.* (1972), Mingle (1977), Civan and Sliepcevich (1983), (1984), Jang *et al.* (1989)) use Bellman's first method to obtain the weighting coefficients because it lets the coordinates of grid points be chosen arbitrarily. Unfortunately, when the order of the algebraic equation system is large, its matrix is ill-conditioned. Thus, it is very difficult to obtain the weighting coefficients for a large number of grid points using this method. To overcome the drawbacks of above methods, Quan and Chang (1989), Wen and Yu (1993a) use Lagrange interpolation polynomials as test functions, and then obtained explicit formulations to determine the weighting coefficients for the first and second order derivatives discretization. More generally, Shu and Richards (1990), and Shu (1991) present the generalized differential quadrature (GDQ), in which all the current methods for determination of weighting coefficients are generalized under the analysis of a high order polynomial approximation and the analysis of a linear vector space. In GDQ, the weighting coefficients of the first order derivative are determined by a simple algebraic formulation without any restriction on choice of grid points, and the weighting coefficients of the second and higher order derivatives are determined by a recurrence relationship. The major advantage of GDQ over DQ is its ease of the computation of the weighting coefficients without any restriction on the choice of grid points. The GDQ method has been successfully applied to some fluid flow problems (see, e.g., Shu (1991), Shu and Richards (1990), (1992a), (1992b)) and structural problems (see, e.g. Du *et al.* (1994), (1995)).

The pioneer work for the application of the DQ method to the general area of structural mechanics was carried out by Bert *et al.* (1988), (1993), (1994), Wang and Bert (1993), and Wang *et al.* (1993), (1994). The recent work of Du *et al.* (1994), (1995), Laura and Gutierrez (1993), and Wen and Yu (1993b) also showed some applications in this area. Like some other numerical methods, the GDQ method discretizes the spatial derivatives and, therefore, reduces the partial differential equations into a set of algebraic equations. To solve these equations, the boundary conditions have to be implemented appropriately. For the case where there is only one boundary condition at each boundary, the implementation is very simple and can be done in a straightforward way. One just needs to replace the discretized governing equations by the boundary conditions at all the boundary points. However, in some cases, there is more than one boundary condition at each boundary, which could result in difficulties in the numerical implementation of the boundary conditions. One example is the solution of the incompressible Navier–Stokes equations. Although the governing equations are second order partial differential equations, there are two boundary conditions for the stream function at each solid boundary, one a Dirichlet type, and other a Neumann type. To implement these two boundary conditions accurately, Shu (1991) proposed an approach which converts the two boundary conditions into two-layer numerical boundary conditions. Another example is the flexural vibration analysis of a thin beam or a plate which will be addressed in the present and companion papers.

The governing equation for bending of a thin beam or a plate is a fourth order differential equation with two boundary conditions at each boundary. Currently, there are various approaches to implement the boundary conditions. One is the so-called δ-technique proposed by Bert *et al.* (1988) and Jang *et al.* (1989), in which two grid points, separated from each other by a small distance δ, are placed near each boundary edge. Then, the two boundary conditions at each boundary are applied at the boundary point itself and its adjacent δ-point. As a result, one boundary condition is exactly satisfied at the boundary while the other (Neumann type) is approximately satisfied at the δ-point. There are two major drawbacks to the δ-technique. One drawback arises from the implementation of one boundary condition at the δ-point. Since it is an approximation to the true boundary condition which should be implemented at the boundary, one can expect that the numerical result is dependent on the choice of the δ-value. To obtain an accurate numerical solution, the δ should be chosen to be very small (possibly not greater than $0.0001l$, where l is the length of the beam or the plate). The small value of δ would cause the second drawback of this technique. When one mesh size (δ) is much smaller than the others, the GDQ weighting coefficient matrices become highly ill-conditioned, which then causes the solution to oscillate. As a result, the numerical solution is less accurate. This has been pointed out by Wang and Bert (1993). The recent work of Wen and Yu (1993b) showed that when two boundary conditions are implemented at one boundary point, then the numerical results are greatly improved. In order to overcome the drawbacks of the δ-technique, Wang and Bert (1993) proposed another approach. In this approach, the derivative boundary conditions are built into the weighting coefficient matrices in the DQ discretization. This approach has been successfully applied to solve some beam and plate problems with very good accuracy. However, as indicated by Wang and Bert (1993), there are some limitations to the application of this approach. One limitation is in the implementation of the clamped–clamped (C–C) type boundary conditions. As will be shown in this paper, the implementation of the

C–C type boundary conditions by this approach leads to some wrong numerical results. Also, for free boundaries of plates, this approach cannot be applied.

In this paper, the approach proposed by Shu (1991) for solving the incompressible Navier–Stokes equations is applied to implement the two boundary conditions at each boundary in the GDQ vibration analysis of beams and plates. As will be seen, this approach removes the drawbacks of the δ-technique. Actually, the major difference between the present approach and the δ-technique is that the two boundary conditions are exactly satisfied in the present approach while only one boundary condition is exactly satisfied in the δ-technique. The implementation of conventional clamped and simply supported boundary conditions in the GDQ vibration analysis of beams and plates is described in the present paper. The implementation of more general boundary conditions including free edges in the GDQ vibration analysis of plates will be addressed in the companion paper (Shu and Du, 1997).

2. GENERALIZED DIFFERENTIAL QUADRATURE

Differential quadrature

Following the concept of classical integral quadrature, Bellman *et al.* (1972) proposed the following approximation

$$f_x(x_i,t) = \sum_{j=1}^{N} c_{ij}^{(1)} \cdot f(x_j,t), \quad \text{for } i = 1,2,\ldots,N, \tag{1}$$

where $f_x(x_i, t)$ indicates the first order derivative of $f(x, t)$ with respect to x at x_i. Obviously, the key procedure in this technique is to determine the weighting coefficients $c_{ij}^{(1)}$. Bellman *et al.* suggested two ways to carry this out. The first way is to let eqn (1) be exact for test functions $g_k(x) = x^k$, $k = 0, 1, \ldots, N-1$, which leads to a set of linear algebraic equations

$$\sum_{j=1}^{N} c_{ij}^{(1)} \cdot x_j^k = k \cdot x_i^{k-1}, \quad \text{for } i = 1,2,\ldots,N; \quad k = 0,1,\ldots,N-1. \tag{2}$$

This equation system has a unique solution because its matrix is of Vandermonde form. Unfortunately, when N is large, the matrix is ill-conditioned and its inversion is difficult. The second way is similar to the first one with an exception that the different test functions

$$g_k(x) = \frac{L_N(x)}{(x-x_k) \cdot L_N^{(1)}(x_k)}, \quad k = 1,2,\ldots,N, \tag{3}$$

are chosen, where $L_N(x)$ is the Nth order Legendre polynomial and $L_N^{(1)}(x)$ the first order derivative of $L_N(x)$. Using the above test functions, Bellman *et al.* (1972) obtained a simple algebraic formulation for calculating $c_{ij}^{(1)}$, but with a condition that the coordinates of grid points should be chosen as the roots of an Nth order Legendre polynomial. Most previous applications of DQ in engineering (see, e.g., Bellman *et al.* (1972), Mingle (1977), Civan and Sliepcevich (1983), (1984), Jang *et al.* (1989)) use Bellman's first method to obtain the weighting coefficients because the grid points can be chosen arbitrarily. However, because of the drawback described above, the number of grid points used is less than or equal to 13. To overcome the drawbacks of DQ, the generalized differential quadrature (GDQ) approach was developed by Shu (1991) for determination of weighting coefficients.

High order polynomial approximation and linear vector space

The GDQ is based on the analysis of a high order polynomial approximation. It is well known that a smooth function in a domain can be approximated by a high order polynomial accurately in accordance with the Weierstrass polynomial approximation theorem. Following this theorem, it is proposed that the solution of a one-dimensional partial differential equation can be approximated by a $(N-1)$th order polynomial.

$$f(x) = \sum_{k=0}^{N-1} a_k \cdot x^k. \tag{4}$$

It is easy to show that the polynomial of degree less than or equal to $N-1$ constitutes an N-dimensional linear vector space V_N. From the concept of linear independence, the bases of a linear vector space can be considered as a linearly independent subset which spans the entire space. Here, if $r_k(x)$, $k = 1, 2, \ldots, N$, are the base polynomials in V_N, $f(x)$ can then be expressed by

$$f(x) = \sum_{k=1}^{N} d_k \cdot r_k(x). \tag{5}$$

Clearly, if all the base polynomials satisfy a linear constrained relationship such as eqn (1), so does $f(x)$. In the linear vector space, there may exist several sets of base polynomials. Each set of base polynomials can be expressed uniquely by another set of base polynomials. It was found that, if the base polynomial $r_k(x)$ is chosen to be x^{k-1}, the same equation system as (2), given by Bellman's first method, can be obtained, and if the base polynomial $r_k(x)$ is taken in the same form as eqn (3), the same formulation, given by Bellman's second method, can be achieved.

Weighting coefficients of first order derivative

For generality, GDQ chooses the base polynomial $r_k(x)$ to be the Lagrange interpolated polynomial

$$r_k(x) = \frac{M(x)}{(x-x_k) \cdot M^{(1)}(x_k)} \tag{6}$$

where

$$M(x) = (x-x_1) \cdot (x-x_2) \ldots (x-x_N)$$
$$M^{(1)}(x_k) = \prod_{j=1, j\neq k}^{N} (x_k - x_j)$$

$x_1, x_2, \ldots, x_N$ are the coordinates of the grid points, and may be chosen arbitrarily.

For simplicity, we set

$$M(x) = N(x, x_k) \cdot (x-x_k), k = 1, 2, \ldots, N \tag{7}$$

with $N(x_i, x_j) = M^{(1)}(x_i) \cdot \delta_{ij}$, where δ_{ij} is the Kronecker operator. Thus, we have

$$M^{(m)}(x) = N^{(m)}(x, x_k) \cdot (x-x_k) + m \cdot N^{(m-1)}(x, x_k)$$
$$\text{for } m = 1, 2, \ldots, N-1;\ k = 1, 2, \ldots, N, \tag{8}$$

where $M^{(m)}(x)$, $N^{(m)}(x, x_k)$ indicate the mth order derivative of $M(x)$ and $N(x, x_k)$. Substituting eqn (6) into eqn (1) and using eqn (8), we obtain

$$c_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j) \cdot M^{(1)}(x_j)}, \quad j \neq i \tag{9a}$$

$$c_{ii}^{(1)} = \frac{M^{(2)}(x_i)}{2M^{(1)}(x_i)}, \quad i = j. \tag{9b}$$

Equation (9) is a simple formulation for computing $c_{ij}^{(1)}$ without any restriction on choice of grid point x_i. Actually, if x_i is given, it is easy to compute $M^{(1)}(x_i)$, thus, $c_{ij}^{(1)}$ for $i \neq j$. The calculation of $c_{ii}^{(1)}$ is based on the computation of the second order derivative $M^{(2)}(x_i)$ which is not easily obtained. As will be shown in the following, this difficulty can be removed by the property of the linear vector space. According to the theory of a linear vector space, one set of base polynomials can be expressed uniquely by another set of base polynomials. Thus, if one set of base polynomials satisfies a linear constrained relationship, say eqn (1), so does another set of base polynomials. Thus, $c_{ij}^{(1)}$ satisfies the following equation which is obtained by the base polynomial x^k when $k = 0$

$$\sum_{j=1}^{M} c_{ij}^{(1)} = 0. \tag{10}$$

From eqn (10), $c_{ii}^{(1)}$ can be easily determined from $c_{ij}^{(1)}$ $(i \neq j)$.

Weighting coefficients of second and higher order derivatives

For the discretization of the second and higher order derivatives, the following linear constrained relationships are applied

$$f_x^{(m)}(x_i, t) = \sum_{j=1}^{N} c_{ij}^{(m)} \cdot f(x_j, t) \tag{11}$$

$$f_x^{(m-1)}(x_i, t) = \sum_{j=1}^{N} c_{ij}^{(m-1)} \cdot f(x_j, t) \quad \text{for } i = 1,2,\ldots,N;\, m = 2,3,\ldots,N-1. \tag{12}$$

Substituting eqn (6) into eqns (11), (12), and using eqns (8) and (9), a recurrence formulation is obtained as follows

$$c_{ij}^{(m)} = m \cdot \left(c_{ij}^{(1)} \cdot c_{ii}^{(m-1)} - \frac{c_{ij}^{(m-1)}}{x_i - x_j} \right), j \neq i \tag{13a}$$

$$c_{ii}^{(m)} = \frac{M^{(m+1)}(x_i)}{(m+1) \cdot M^{(1)}(x_i)}, j = i \tag{13b}$$

$$\text{for } i,j = 1,2,\ldots,N;\, m = 2,3,\ldots,N-1,$$

where $c_{ij}^{(1)}$ are the weighting coefficients of the first order derivative described above. Again, in terms of the analysis of the N-dimensional linear vector space, the equation system for $c_{ij}^{(m)}$ derived from the Lagrange interpolated polynomials should be equivalent to that derived from the base polynomials x^k, $k = 0, 1, \ldots, N-1$. Thus, $c_{ij}^{(m)}$ should satisfy the following equation obtained from the base polynomial x^k when $k = 0$

$$\sum_{j=1}^{N} c_{ij}^{(m)} = 0. \tag{14}$$

From this formulation, $c_{ii}^{(m)}$ can be determined from $c_{ij}^{(m)}$ $(i \neq j)$.

Extension to the multi-dimensional case

For the two-dimensional approximation of a function $f(x, y)$ in a rectangular domain, it is supposed that the value of $f(x, b)$, where b is a constant, can be approximated by a $(N-1)$th order polynomial $P_N(x)$ which constitutes an N-dimensional linear vector space V_N with N base polynomials $r_i(x)$, $i = 1, 2, \ldots, N$, and the value of $f(a, y)$, where a is a constant, can be approximated by a $(M-1)$th order polynomial $P_M(y)$ which constitutes a

M-dimensional linear vector space V_M with M base polynomials $s_j(y), j = 1,2,\ldots,M$. The value of function $f(x,y)$ can be approximated by the polynomial $Q_{N\times M}(x,y)$ with the form

$$Q_{N\times M}(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}\cdot x^{i-1}\cdot y^{j-1} \tag{15}$$

where a_{ij} is a coefficient. It is obvious that $Q_{N\times M}(x,y)$ constitutes an $N\times M$ dimensional linear polynomial vector space $V_{N\times M}$. In the following, it is shown that $\Phi_{ij}(x,y) = r_i(x)\cdot s_j(y)$ constitutes the base polynomials in the vector space $V_{N\times M}$. Since $r_i(x)$, $s_j(y)$ are the base polynomials of V_N and V_M, they must be linearly independent, that is

$$\sum_{i=1}^{N} b_i\cdot r_i(x) = 0 \quad \text{only if } b_i = 0, i = 1,2,\ldots,N. \tag{16}$$

$$\sum_{j=1}^{M} d_j\cdot s_j(y) = 0 \quad \text{only if } d_j = 0, j = 1,2,\ldots,M. \tag{17}$$

It is observed that if

$$\sum_{i=1}^{N}\sum_{j=1}^{M} e_{ij}\cdot\Phi_{ij}(x,y) = 0, \text{i.e. } \sum_{i=1}^{N}\left[\sum_{j=1}^{M} e_{ij}\cdot s_j(y)\right]\cdot r_i(x) = 0 \tag{18}$$

then from eqn (16), the following equation can be obtained

$$\sum_{j=1}^{M} e_{ij}\cdot s_j(y) = 0.$$

Finally, from eqn (17), we obtain $e_{ij} = 0$. Thus, $\Phi_{ij}(x,y)$ constitutes the base polynomials in $V_{N\times M}$.

Now, it is assumed that the following constrained relations are satisfied for function $f(x,y,t)$ and its first order spatial derivatives

$$f_x(x_i,y_j,t) = \sum_{k=1}^{N} c_{ik}^{(1)}\cdot f(x_k,y_j,t) \tag{19}$$

$$f_y(x_i,y_j,t) = \sum_{k=1}^{M} \bar{c}_{jk}^{(1)}\cdot f(x_i,y_k,t)$$

$$\text{for } i = 1,2,\ldots,N; \quad j = 1,2,\ldots,M, \tag{20}$$

where $c_{ik}^{(1)}$, $\bar{c}_{jk}^{(1)}$ are the weighting coefficients related to $f_x(x_i,y_j,t)$ and $f_y(x_i,y_j,t)$, respectively. If all the base polynomials $\Phi_{ij}(x,y)$ satisfy eqns (19), (20), then so does any polynomial in $V_{N\times M}$. Substituting $\Phi_{ij}(x,y)$ into eqns (19), (20) yields

$$\sum_{k=1}^{N} c_{ik}^{(1)}\cdot r_j(x_k) = r_j^{(1)}(x_i) \tag{21}$$

$$\sum_{k=1}^{M} \bar{c}_{ik}^{(1)}\cdot s_j(y_k) = s_j^{(1)}(y_i) \tag{22}$$

where $r_j^{(1)}(x_i)$ represents the first order derivative of $r_j(x)$ at x_i and $s_j^{(1)}(y_i)$ represents the first order derivative of $s_j(y)$ at y_i. From eqns (21), (22), it is obvious that $c_{ik}^{(1)}$ or $\bar{c}_{jk}^{(1)}$ is only related to $r_i(x)$ or $s_j(y)$. Hence, the formulation of the one dimensional case can be directly extended to the two dimensional case.

3. IMPLEMENTATION OF BOUNDARY CONDITIONS

In this section, the GDQ method is applied to analyze the free vibrations of beams and rectangular plates. A new methodology for implementing various conventional boundary conditions is presented, which directly substitutes the boundary conditions into the governing equations (SBCGE) and then simplifies the resultant eigenvalue equation system.

Vibration of beams

The non-dimensional governing equation for the free vibration of a Bernoulli–Euler beam of varying cross-section may be written as

$$s(X)\cdot\frac{\mathrm{d}^4W}{\mathrm{d}X^4}+2\frac{\mathrm{d}s(X)}{\mathrm{d}X}\cdot\frac{\mathrm{d}^3W}{\mathrm{d}X^3}+\frac{\mathrm{d}^2s(X)}{\mathrm{d}X^2}\cdot\frac{\mathrm{d}^2W}{\mathrm{d}X^2}-\Omega^2\cdot W=0 \tag{23}$$

where

$$s(X)=\frac{EI}{EI_0},\Omega^2=\frac{\rho AL^4}{EI_0}\omega^2,X=\frac{x}{L},$$

EI is the beam's flexural rigidity, ρA is the mass per unit length, L is the length of the beam, ω is the dimensional frequency. For a beam of varying cross-section, EI and A are functions of the coordinate x. Equation (23) is a 4th order ordinary differential equation. For a well-posed problem, it requires four boundary conditions. These can be given by specifying two boundary conditions at the end $X = 0$, and another two boundary conditions at the end $X = 1$. In the present work, the following two types of boundary conditions are considered: Simply supported end (SS)

$$W=0 \quad \text{and} \quad \frac{\partial^2W}{\partial X^2}=0 \tag{24a}$$

Clamped end (C)

$$W=0 \quad \text{and} \quad \frac{\partial W}{\partial X}=0. \tag{24b}$$

For the numerical computation, the continuous solution is approximated by the function values at discrete points. Now, we assume that the computational domain $0 \leqslant X \leqslant 1$ is divided by $(N-1)$ intervals with coordinates of grid points as $X_1, X_2, \ldots, X_N$. With the coordinates of grid points, the GDQ weighting coefficients can be computed through eqns (9a), (10), (13a) and (14). Then, applying the GDQ method to discretize the spatial derivatives of eqn (23) yields

$$s^{(2)}(X_i)\cdot\sum_{k=1}^{N}c_{ik}^{(2)}\cdot W_k+2\cdot s^{(1)}(X_i)\cdot\sum_{k=1}^{N}c_{ik}^{(3)}\cdot W_k+s(X_i)\cdot\sum_{k=1}^{N}c_{ik}^{(4)}\cdot W_k=\Omega^2\cdot W_i \tag{25}$$

where W_i, $i = 1, 2, \ldots, N$, is the functional value at the grid point X_i, $s^{(2)}(X_i), s^{(1)}(X_i)$ are the second and first order derivatives of $s(X)$ at X_i. Similarly, the derivatives in the boundary conditions can be discretized by the GDQ method. As a result, the numerical boundary conditions can be written as

$$W_1 = 0 \tag{26a}$$

$$\sum_{k=1}^{N} c_{1,k}^{(n0)} \cdot W_k = 0 \tag{26b}$$

$$W_N = 0 \tag{26c}$$

$$\sum_{k=1}^{N} c_{N,k}^{(n1)} \cdot W_k = 0 \tag{26d}$$

where $n0$, $n1$ may be taken as either 1 or 2. By choosing $n0$, $n1$, eqn (26) can give the following four sets of boundary conditions,

$n0 = 1, n1 = 1$—clamped–clamped
$n0 = 1, n1 = 2$—clamped–simply supported
$n0 = 2, n1 = 1$—simply supported–clamped
$n0 = 2, n1 = 2$—simply supported–simply supported.

Equations (26a) and (26c) can be easily substituted into equation system (25). This is not the case for eqns (26b) and (26d). However, one can couple these two eqns (26b), (26d) together to give two solutions, W_2 and W_{N-1}, as

$$W_2 = \frac{1}{AXN} \cdot \sum_{k=3}^{N-2} AXK1 \cdot W_k \tag{27a}$$

$$W_{N-1} = \frac{1}{AXN} \cdot \sum_{k=3}^{N-2} AXKN \cdot W_k \tag{27b}$$

where

$$AXK1 = c_{1,k}^{(n0)} \cdot c_{N,N-1}^{(n1)} - c_{1,N-1}^{(n0)} \cdot c_{N,k}^{(n1)}$$

$$AXKN = c_{1,2}^{(n0)} \cdot c_{N,k}^{(n1)} - c_{1,k}^{(n0)} \cdot c_{N,2}^{(n1)}$$

$$AXN = c_{N,2}^{(n1)} \cdot c_{1,N-1}^{(n0)} - c_{1,2}^{(n0)} \cdot c_{N,N-1}^{(n1)}.$$

According to eqn (27), W_2 and W_{N-1} are expressed in terms of $W_3, W_4, \ldots, W_{N-2}$, and can be easily substituted into the equation system (25). It is noted that eqn (26) provides four boundary equations. In total, we have N unknowns $W_1, \ldots, W_N$. So, to close the system, the discretized governing eqn (25) has to be applied at $(N-4)$ grid points. This can be done by applying eqn (25) at grid points $X_3, X_4, \ldots, X_{N-2}$. Substituting eqns (26a), (26c), (27) into eqn (25) gives

$$s^{(2)}(X_i) \cdot \sum_{k=3}^{N-2} C_1 \cdot W_k + 2 \cdot s^{(1)}(X_i) \cdot \sum_{k=3}^{N-2} C_2 \cdot W_k + s(X_i) \cdot \sum_{k=3}^{N-2} C_3 \cdot W_k = \Omega^2 \cdot W_i$$

$$\text{for } i = 3, 4, \ldots, N-2. \tag{28}$$

where

$$C_1 = c_{i,k}^{(2)} - \frac{c_{i,2}^{(2)} \cdot AXK1 + c_{i,N-1}^{(2)} \cdot AXKN}{AXN}$$

$$C_2 = c_{i,k}^{(3)} - \frac{c_{i,2}^{(3)} \cdot AXK1 + c_{i,N-1}^{(3)} \cdot AXKN}{AXN}$$

$$C_3 = c_{i,k}^{(4)} - \frac{c_{i,2}^{(4)} \cdot AXK1 + c_{i,N-1}^{(4)} \cdot AXKN}{AXN}.$$

It is noted that equation system (28) has $(N-4)$ equations and $(N-4)$ unknowns, which can be written in an eigenvalue matrix form

$$[A]\{W\} = \Omega^2\{W\} \tag{29}$$

where

$$\{W\} = \{W_3, W_4, \ldots, W_{N-2}\}^T$$

It is noted that the δ-technique discussed earlier also uses four boundary condition equations similar to eqn (26). Equations (26a) and (26c) are the same in the δ-technique. However, eqns (26b) and (26d) are replaced in the δ-technique by

$$\sum_{k=1}^{N} c_{2,k}^{(n0)} \cdot W_k = 0$$

$$\sum_{k=1}^{N} c_{N-1,k}^{(n1)} \cdot W_k = 0.$$

X_2 and X_{N-1} are chosen as δ and $1-\delta$ in the δ-technique. Obviously, the δ-technique does not implement the derivative boundary conditions at the correct positions.

Vibration of thin rectangular plates

The non-dimensional differential equation for a thin uniform thickness, rectangular plate may be written as

$$\frac{\partial^4 W}{\partial X^4} + 2\lambda^2 \frac{\partial^4 W}{\partial X^2 \partial Y^2} + \lambda^4 \frac{\partial^4 W}{\partial Y^4} = \Omega^2 \cdot W \tag{30}$$

where W is the dimensionless mode function; Ω is the dimensionless frequency; $X = x/a$, $Y = y/b$ are dimensionless coordinates, a and b are the lengths of the plate edges; $\lambda = a/b$ is the aspect ratio. Further, $\Omega = \omega a^2\sqrt{\rho/D}$, where ω is the dimensional circular frequency, $D = Eh^3/[12(1-\nu^2)]$ is the flexural rigidity, E, ν, ρ and h are Young's modulus, Poisson's ratio, density of the plate material, and the plate thickness, respectively. Equation (30) is a 4th order partial differential equation with respect to X and Y. Thus, it requires two boundary conditions at each edge. The following two types of boundary conditions are considered.

Simply supported edge (SS)

$$W = 0, \frac{\partial^2 W}{\partial X^2} = 0 \tag{31a}$$

at $X = 0$ or $X = 1$, and

$$W = 0, \frac{\partial^2 W}{\partial Y^2} = 0 \tag{31b}$$

at $Y = 0$ or $Y = 1$.

Clamped edge (C)

$$W = 0, \frac{\partial W}{\partial X} = 0 \tag{32a}$$

at $X = 0$ or $X = 1$, and

$$W = 0, \frac{\partial W}{\partial Y} = 0 \tag{32b}$$

at $Y = 0$ or $Y = 1$.

By applying the GDQ method, eqn (30) can be discretized as

$$\sum_{k=1}^{N} c_{i,k}^{(4)} \cdot W_{k,j} + 2\lambda^2 \cdot \sum_{k1=1}^{N} \sum_{k2=1}^{M} c_{i,k1}^{(2)} \cdot \bar{c}_{j,k2}^{(2)} \cdot W_{k1,k2} + \lambda^4 \cdot \sum_{k=1}^{M} \bar{c}_{j,k}^{(4)} \cdot W_{i,k} = \Omega^2 \cdot W_{i,j} \tag{33}$$

where N, M are the number of grid points in the X and Y directions, respectively, and $c_{i,k}^{(n)}$, $\bar{c}_{j,k}^{(m)}$ are the weighting coefficients in the X and Y directions. Similarly, using the GDQ approach, the boundary conditions (31), (32) can be generalized as

$$W_{1,j} = 0, \quad W_{N,j} = 0, \quad W_{i,1} = 0, \quad W_{i,M} = 0 \quad \text{for } i = 1, 2, \ldots, N; j = 1, 2, \ldots, M, \tag{34a}$$

$$\sum_{k=1}^{N} c_{1,k}^{(n0)} \cdot W_{k,j} = 0 \quad \text{for } j = 2, 3, \ldots, M-1, \tag{34b}$$

$$\sum_{k=1}^{N} c_{N,k}^{(n1)} \cdot W_{k,j} = 0 \quad \text{for } j = 2, 3, \ldots, M-1, \tag{34c}$$

$$\sum_{k=1}^{M} \bar{c}_{1,k}^{(m0)} \cdot W_{i,k} = 0 \quad \text{for } i = 2, 3, \ldots, N-1, \tag{34d}$$

$$\sum_{k=1}^{M} \bar{c}_{M,k}^{(m1)} \cdot W_{i,k} = 0 \quad \text{for } i = 2, 3, \ldots, N-1, \tag{34e}$$

where $n0$, $n1$, $m0$, $m1$ are taken as either 1 or 2 : 1 is for the clamped edge condition and 2 is for the simply supported edge condition. $n0$, $n1$, $m0$, $m1$ are for the edges of $X = 0$, $X = 1$, $Y = 0$, $Y = 1$, respectively. It is noted that eqn (34a) corresponds to the Dirichlet boundary conditions at the four edges of the plate, and eqns (34b), (34c), (34d) and (34e) result from the derivative boundary conditions. Obviously, eqn (34a) can be easily substituted into the equation system (33). However, eqns (34b), (34c), (34d) and (34e) cannot be directly put into the equation system (33). This difficulty can be eliminated by the following approach. Using the same fashion as for beams, eqns (34b), (34c) can be coupled to give two solutions, $W_{2,j}$, $W_{N-1,j}$, which are located at the grid points shown by the symbol ○ in Fig. 1, as

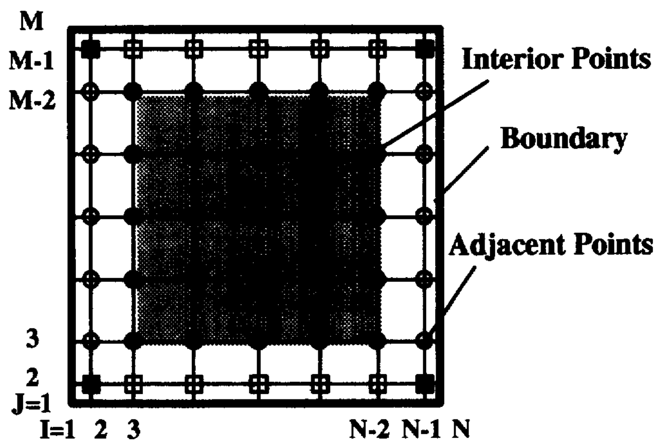

Fig. 1. Illustration of interior and adjacent points for a rectangular plate.

$$W_{2,j} = \frac{1}{AXN} \cdot \sum_{k=3}^{N-2} AXK1 \cdot W_{k,j} \tag{35a}$$

$$W_{N-1,j} = \frac{1}{AXN} \cdot \sum_{k=3}^{N-2} AXKN \cdot W_{k,j} \quad \text{for } j = 3, 4, \ldots, M-2, \tag{35b}$$

where

$$AXN = c_{N,2}^{(n1)} \cdot c_{1,N-1}^{(n0)} - c_{1,2}^{(n0)} \cdot c_{N,N-1}^{(n1)}$$

$$AXK1 = c_{1,k}^{(n0)} \cdot c_{N,N-1}^{(n1)} - c_{1,N-1}^{(n0)} \cdot c_{N,k}^{(n1)}$$

$$AXKN = c_{1,2}^{(n0)} \cdot c_{N,k}^{(n1)} - c_{1,k}^{(n0)} \cdot c_{N,2}^{(n1)}.$$

Similarly, eqns (34d), (34e) can be coupled to give two solutions, W_{i2}, $W_{i,M-1}$, which are located at the grid points shown by the symbol □ in Fig. 1, as

$$W_{i,2} = \frac{1}{AYM} \cdot \sum_{k=3}^{M-2} AYK1 \cdot W_{i,k} \tag{36a}$$

$$W_{i,M-1} = \frac{1}{AYM} \cdot \sum_{k=3}^{M-2} AYKM \cdot W_{i,k} \quad \text{for } i = 3, 4, \ldots, N-2, \tag{36b}$$

where

$$AYM = \bar{c}_{M,2}^{(m1)} \cdot \bar{c}_{1,M-1}^{(m0)} - \bar{c}_{1,2}^{(m0)} \cdot \bar{c}_{M,M-1}^{(m1)}$$

$$AYK1 = \bar{c}_{1,k}^{(m0)} \cdot \bar{c}_{M,M-1}^{(m1)} - \bar{c}_{1,M-1}^{(m0)} \cdot \bar{c}_{M,k}^{(m1)}$$

$$AYKM = \bar{c}_{1,2}^{(m0)} \cdot \bar{c}_{M,k}^{(m1)} - \bar{c}_{1,k}^{(m0)} \cdot \bar{c}_{M,2}^{(m1)}.$$

For the points near the four corners shown by the symbol ■ in Fig. 1, the four eqns (34b), (34c), (34d), (34e) have to be coupled to provide the following four solutions

$$W_{2,2} = \frac{1}{AXN} \cdot \frac{1}{AYM} \cdot \sum_{k1=3}^{N-2} \sum_{k2=3}^{M-2} AXK1 \cdot AYK1 \cdot W_{k1,k2} \tag{37a}$$

$$W_{N-1,2} = \frac{1}{AXN} \cdot \frac{1}{AYM} \cdot \sum_{k1=3}^{N-2} \sum_{k2=3}^{M-2} AXKN \cdot AYK1 \cdot W_{k1,k2} \tag{37b}$$

$$W_{2,M-1} = \frac{1}{AXN} \cdot \frac{1}{AYM} \cdot \sum_{k1=3}^{N-2} \sum_{k2=3}^{M-2} AXK1 \cdot AYKM \cdot W_{k1,k2} \tag{37c}$$

$$W_{N-1,M-1} = \frac{1}{AXN} \cdot \frac{1}{AYM} \cdot \sum_{k1=3}^{N-2} \sum_{k2=3}^{M-2} AXKN \cdot AYKM \cdot W_{k1,k2}. \tag{37d}$$

For eqn (37), the index k in the $AXK1$ and $AXKN$ expressions is replaced by $k1$, and the index k in the $AYK1$ and $AYKM$ expressions is replaced by $k2$. With eqns (34a), (35), (36) and (37), all the boundary conditions can be directly substituted into the governing equation system (33). As a result, the final eigenvalue equation system becomes

$$\sum_{k=3}^{N-2} \cdot C_1 \cdot W_{k,j} + 2\lambda^2 \cdot \sum_{k1=3}^{N-2} \sum_{k2=3}^{M-2} C_2 \cdot W_{k1,k2} + \lambda^4 \cdot \sum_{k=3}^{M-2} \cdot C_3 \cdot W_{i,k} = \Omega^2 \cdot W_{i,j} \tag{38}$$

where

$$C_1 = c_{i,k}^{(4)} - \frac{c_{i,2}^{(4)} \cdot AXK1 + c_{i,N-1}^{(4)} \cdot AXKN}{AXN},$$

$$C_2 = c_{i,k1}^{(2)} \cdot \bar{c}_{j,k2}^{(2)} - \frac{(AXK1 \cdot c_{i,2}^{(2)} + AXKN \cdot c_{i,N-1}^{(2)})}{AXN} \cdot \bar{c}_{j,k2}^{(2)}$$

$$- \frac{(AYK1 \cdot \bar{c}_{j,2}^{(2)} + AYKM \cdot \bar{c}_{j,M-1}^{(2)})}{AYM} \cdot c_{i,k1}^{(2)}$$

$$+ \frac{(AXK1 \cdot AYK1 \cdot c_{i,2}^{(2)} \cdot \bar{c}_{j,2}^{(2)} + AXKN \cdot AYK1 \cdot c_{i,N-1}^{(2)} \cdot \bar{c}_{j,2}^{(2)})}{AXN \cdot AYM}$$

$$+ \frac{(AXK1 \cdot AYKM \cdot c_{i,2}^{(2)} \cdot \bar{c}_{j,M-1}^{(2)} + AXKN \cdot AYKM \cdot c_{i,N-1}^{(2)} \cdot \bar{c}_{j,M-1}^{(2)})}{AXN \cdot AYM}$$

$$C_3 = \bar{c}_{j,k}^{(4)} - \frac{\bar{c}_{j,2}^{(4)} \cdot AYK1 + \bar{c}_{j,M-1}^{(4)} \cdot AYKM}{AYM}.$$

Since the equation system (38) has $(N-4) \times (M-4)$ unknowns, it should be applied at $(N-4) \times (M-4)$ interior points to close the system. This can be done by applying eqn (38) at the interior points $3 \leqslant i \leqslant N-2, 3 \leqslant j \leqslant M-2$ as shown in Fig. 1. Similarly, eqn (38) can be put in a matrix form

$$[A]\{W\} = \Omega^2\{W\} \tag{39}$$

where

$$\{W\} = \{W_{3,3}, \ldots, W_{3,M-2}, W_{4,3}, \ldots, W_{4,M-2}, \ldots, W_{N-2,3}, \ldots, W_{N-2,M-2}\}^T.$$

4. RESULTS AND DISCUSSION

As mentioned earlier, the present approach overcomes the drawbacks of the δ-technique in implementing the derivative conditions. Thus, the δ-technique will not be used to validate the present approach. The method of modifying the weighting coefficient matrices (MWCM) proposed by Wang and Bert (1993) is different from the present approach. The idea of the MWCM is to directly incorporate the zero derivative boundary conditions into the weighting coefficient matrices. The comparison of the present approach and the MWCM method will be discussed through their applications to the vibration analysis of beams and plates with combinations of simply supported and clamped boundary conditions.

In the present study, the coordinates of the grid points for the beam are chosen as

$$X_i = \frac{1-\cos\left(\frac{i-1}{N-1} \cdot \pi\right)}{2}, \quad i = 1, 2, \ldots, N. \tag{40}$$

Numerical computations of natural frequencies for a uniform beam $s(X) = 1$ are conducted under three sets of boundary conditions, that is, simply supported–simply supported (SS–SS), clamped–clamped (C–C), and clamped–simply supported (C–SS) conditions. The GDQ results using the present approach for implementing the boundary conditions are compared with the well known exact solutions (cf. Belvins (1984)) and the GDQ results

Table 1. Comparison of natural frequencies of a uniform beam (N = 15)

	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
		simply supported–simply supported			
Blevins (1984)	9.8696	39.4784	88.8264	157.9137	246.7401
SBCGE	9.8696	39.4784	88.8249	158.0619	248.4716
MWCM	9.8696	39.4784	88.8264	157.9138	246.7409
		clamped–clamped			
Blevins (1984)	22.3733	61.6728	120.9034	199.8594	298.5555
SBCGE	22.3733	61.6728	120.9021	199.9365	299.3886
MWCM	0.012518	22.3733	61.6728	120.8963	199.9665
		clamped–simply supported			
Blevins (1984)	15.4182	49.9648	104.2477	178.2697	272.0310
SBCGE	15.4182	49.9648	104.2471	178.4642	273.1126
MWCM	15.4182	49.9648	104.2477	178.2671	271.9485

using the MWCM method. Table 1 lists natural frequencies of the first 5 modes for the above-mentioned three sets of boundary conditions. Included in Table 1 are the SBCGE results, MWCM results, and the exact solutions (cf. Blevins (1984)). The GDQ results are obtained using 15 grid points. It can be observed from Table 1 that, for the SS–SS boundary condition, although the SBCGE and the MWCM results agree very well with the exact solutions, the MWCM method gives better accuracy than the SBCGE approach. Actually, the SBCGE method requires 17 grid points to reach the same accuracy as the MWCM method with the use of 15 grid points. This phenomenon can be analyzed as follows. The error of numerical results to the true solution of a partial differential equation is due to the truncated error arising from the numerical discretization of derivatives in the partial differential equation and the boundary conditions. In the MWCM method, the derivative conditions are built into the weighting coefficient matrices, which are exactly satisfied without any numerical discretization while in the SBCGE approach, the derivatives in both the partial differential equation and the boundary conditions are discretized by GDQ with a high order of accuracy. So, we can see that the numerical errors in the MWCM method are only contributed by the discretization of the derivatives in the partial differential equation while the numerical errors in the SBCGE approach are contributed by the discretization of derivatives in both the partial differential equation and the boundary conditions. It can be expected that the MWCM approach provides less numerical errors than the SBCGE approach. However, for the C–C boundary condition, the MWCM results are partially wrong. In contrast, the SBCGE method works uniformly well for both the SS–SS and the C–C boundary conditions. As discussed earlier, there are 4 boundary conditions for beams, and for numerical computation, there are N unknowns lying at the N grid points. So, to close the system, the discretization of the governing equation should only be applied at $(N-4)$ interior grid points, leading to a dimension of the resultant eigenvalue equation system as $(N-4)$. The SBCGE approach exactly follows this procedure. In the MWCM method, however, the derivative conditions are built into the weighting coefficient matrices, and only two Dirichlet conditions need to be implemented. Therefore, the discretization of the governing equation has to be applied at $(N-2)$ interior points to close the system. As a result, the dimension of the resultant MWCM eigenvalue equation system is $(N-2)$ instead of $(N-4)$. In other words, the MWCM method provides two more frequencies than the SBCGE approach. For the SS–SS beam, these two additional frequencies are high frequencies, which do not affect the true natural frequency distribution. However, for the C–C boundary condition, one of the two additional frequencies is very small, which causes the computed results to be wrongly distributed. In fact, it can be observed from Table 1 that, for the MWCM results of C–C boundary condition, if the first frequency is removed, the remaining frequencies match the exact solution very well. The results of the C–SS boundary condition are also shown in Table 1. For this case, the MWCM method provides more accurate results than the SBCGE method since the two additional frequencies are high frequencies. It seems that if the additional frequencies are high frequencies, the MWCM method could provide more accurate results than the SBCGE method. The accuracy of the SBCGE results can be improved by using more grid points.

For the rectangular plate, the X-coordinates are taken from eqn (40), and the Y-coordinates are chosen as

$$Y_j = \frac{1-\cos\left(\frac{j-1}{M-1}\cdot\pi\right)}{2}, \quad j = 1,2,\ldots,M. \tag{41}$$

For the free vibration of rectangular plates, there are a variety of publications available. Among these, the work of Leissa (1973) is most complete in that it presents the frequency data of all twenty-one plate configurations for the first nine modes and for a wide range of aspect ratios. Thus, Leissa's results are chosen to evaluate the accuracy of the SBCGE results. For the rectangular plate, the frequency data are obtained for aspect ratios of $\lambda = a/b = 2/5$, $2/3$, 1, $3/2$, $5/2$, using 12 grid points in both the X and Y directions. Table 2 shows the natural frequencies of the first 5 modes with all four edges simply supported. The SBCGE, MWCM, and Leissa's results are included in the table. For this case, Leissa's results are the exact solutions. It can be observed that, by comparing with the exact solution, the MWCM method provides better accuracy than the SBCGE method. The reason is the same as for the beam. It is noted that, for the rectangular plate, the dimension of the resultant MWCM eigenvalue equation system is $(N-2)\times(M-2)$ while the dimension of the resultant SBCGE eigenvalue equation system is $(N-4)\times(M-4)$. Thus, the MWCM method provides $(2N+2M-12)$ more frequencies than the SBCGE method. For the SS–SS–SS–SS boundary conditions, these additional frequencies are high frequencies, which do not affect the true natural frequency distribution. However, for the C–C–C–C boundary conditions, some of these frequencies are low frequencies, which then distort the true natural frequency distribution of the problem. This can be seen clearly in Table 3. The MWCM results in Table 3 are completely wrong. As discussed earlier, by using the MWCM method for the beam, there are two additional frequencies. However, for the rectangular plate, there are $(2N+2M-12)$ additional frequencies, which depend on the number of grid points N and M. Some of these additional frequencies are low frequencies which can result in the wrong frequency distribution. We can expect that the number of such low frequencies is also dependent on the number of grid points N and M. Thus, it is difficult to know the actual number of such low frequencies which can distort the true natural frequency distribution. For the case where one pair of two opposite edges are simply supported and

Table 2. Natural frequencies of a rectangular plate ($N = M = 12$, SS–SS–SS–SS)

$\lambda = a/b$		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/5	Leissa (1973)	11.4487	16.1862	24.0818	35.1358	41.0576
	SBCGE	11.4487	16.1826	24.0529	34.7326	41.0592
	MWCM	11.4493	16.1859	24.0818	35.1319	41.0576
2/3	Leissa (1973)	14.2561	27.4156	43.8649	49.3480	57.0244
	SBCGE	14.2561	27.4161	43.8665	49.2377	57.0258
	MWCM	14.2561	27.4156	43.8649	49.3478	57.0244
1	Leissa (1973)	19.7392	49.3480	49.3480	78.9568	98.6960
	SBCGE	19.7392	49.3495	49.3495	78.9586	98.4154
	MWCM	19.7369	49.3480	49.3480	78.9568	98.6956
3/2	Leissa (1973)	32.0762	61.6850	98.6960	111.0330	128.3049
	SBCGE	32.0762	61.6861	98.6996	110.7848	128.3081
	MWCM	32.0762	61.6850	98.6960	111.0326	128.3049
5/2	Leissa (1973)	71.5564	101.1634	150.5115	219.5987	256.6097
	SBCGE	71.5546	101.1641	150.3305	217.0789	256.6205
	MWCM	71.5582	101.1620	150.5110	219.5742	256.6100

Table 3. Natural frequencies of a rectangular plane ($N = M = 12$, C–C–C–C)

$\lambda = a/b$		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/5	Leissa (1973)	23.648	27.817	35.446	46.702	61.554
	SBCGE	23.645	27.810	35.414	46.451	62.091
	MWCM	0.215	3.579	9.873	19.349	22.390
2/3	Leissa (1973)	27.010	41.716	66.143	66.552	79.850
	SBCGE	27.006	41.707	66.130	66.465	79.819
	MWCM	0.477	9.945	22.387	27.005	27.350
1	Leissa (1973)	35.992	73.413	73.413	108.270	131.640
	SBCGE	35.986	73.399	73.399	108.230	131.418
	MWCM	0.788	22.381	22.381	35.985	61.368
3/2	Leissa (1973)	60.772	93.860	148.820	149.740	179.660
	SBCGE	60.763	93.841	148.793	149.547	179.594
	MWCM	1.074	22.376	50.370	60.761	61.537
5/2	Leissa (1973)	147.800	173.850	221.540	291.890	384.710
	SBCGE	147.779	173.815	221.338	290.321	388.317
	MWCM	1.343	22.371	61.706	120.932	139.940

the other pair of two opposite edges are clamped, the MWCM results are also wrong. This can be seen in Table 4. Compared with the results of the C–C–C–C case, the MWCM results show some improvement for the SS–C–SS–C case. This is because some low additional frequencies of the C–C–C–C case shift to high additional frequencies in the SS–C–SS–C case. If the clamped boundary condition is not imposed at the two opposite edges, then the MWCM method provides accurate numerical results. This can be observed in Table 5, which shows the natural frequencies of the first 5 modes for the C–C–SS–SS case. For all the cases, the SBCGE method is found to work uniformly well. Tables 2–5 demonstrate that, by using the same number of grid points ($N = M = 12$), the SBCGE results have the same order of accuracy for the SS–SS–SS–SS, C–C–C–C, SS–C–SS–C, C–C–SS–SS cases. The accuracy of these results can be improved by increasing the number of grid points.

5. CONCLUSIONS

In this paper, a new approach is proposed to implement the conventional boundary conditions in the GDQ free vibration analysis of beams and rectangular plates. In the proposed approach, the derivative conditions for the two opposite edges $i = 1$ and $i = N$

Table 4. Natural frequencies of a rectangular plate ($N = M = 12$, SS–C–SS–C)

$\lambda = a/b$		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/5	Leissa (1973)	12.135	18.365	27.966	40.750	41.378
	SBCGE	12.135	18.366	27.954	40.483	41.385
	MWCM	9.872	12.135	18.373	27.968	39.538
2/3	Leissa (1973)	17.373	35.345	45.429	62.054	62.313
	SBCGE	17.373	35.346	45.433	61.987	62.321
	MWCM	9.870	17.373	35.352	39.494	45.429
1	Leissa (1973)	28.951	54.743	69.327	94.585	102.216
	SBCGE	28.951	54.745	69.329	94.589	101.950
	MWCM	9.870	28.951	39.482	54.743	69.344
3/2	Leissa (1973)	56.348	78.984	123.172	146.268	170.111
	SBCGE	56.348	78.984	122.950	146.273	170.117
	MWCM	9.870	39.479	56.348	78.984	88.833
5/2	Leissa (1973)	145.484	164.739	202.227	261.105	342.144
	SBCGE	145.484	164.739	202.093	260.890	340.485
	MWCM	9.870	39.478	88.827	145.484	157.894

Table 5. Natural frequencies of a rectangular plate ($N = M = 12$, C–C–SS–SS)

$\lambda = a/b$		Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
2/5	Leissa (1973)	16.849	21.363	29.236	40.509	51.457
	SBCGE	16.848	21.357	29.201	40.207	51.454
	MWCM	16.847	21.357	29.225	40.515	51.449
2/3	Leissa (1973)	19.952	34.024	54.370	57.517	67.815
	SBCGE	19.952	34.020	54.366	57.376	67.790
	MWCM	19.951	34.020	54.363	57.510	67.788
1	Leissa (1973)	27.056	60.544	60.791	92.865	114.570
	SBCGE	27.054	60.540	60.788	92.834	114.202
	MWCM	27.054	60.538	60.786	92.835	114.563
3/2	Leissa (1973)	44.893	76.554	122.330	129.410	152.580
	SBCGE	44.891	76.545	122.324	129.096	152.527
	MWCM	44.890	76.544	122.317	129.398	152.522
5/2	Leissa (1973)	105.310	133.520	182.730	253.180	321.600
	SBCGE	105.200	133.484	182.505	253.295	321.587
	MWCM	105.296	133.480	182.657	253.222	321.556

are coupled to provide two solutions at two neighboring points to the edge $i = 2$ and $i = N-1$, which are expressed in terms of the function values at interior points $i = 3, 4, \ldots, N-2$ (see Fig. 1). These solutions are then substituted into the governing equations. The present approach overcomes the drawbacks of the previous δ-technique in implementing the derivative conditions. The δ-technique discretizes the derivative boundary conditions at a point of distance δ away from the boundary while the present approach discretizes the derivative boundary condition exactly at the boundary point itself. Compared with the method of modifying weighting coefficient matrices (MWCM), the dimension of the resultant eigenvalue equation system in the present approach is reduced by 2 for the beam, and $(2N+2M-12)$ for the rectangular plate. The present approach was compared with the MWCM method through their applications to the beam and the rectangular plate with simply supported and clamped boundary conditions. It was found that, for simply supported boundary conditions, the MWCM method gives more accurate numerical results. However, for the case where the opposite edges are clamped–clamped, the MWCM results are incorrect. The present approach works uniformly well for any combination of simply supported and clamped boundary conditions.

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